Random Matrix Theory for MIMO Communications

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Invited Seminar

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Technische Universität München (TUM)
Technical University of Munich

- Established in 1868
- Located in Munich, Bavaria, Germany
- 12 departments → Electrical Engineering and Information Technology
- Approximately 21,600 students (2006)
- 4,160 academic staff-395 Professors
- Consistent ranking amongst the best universities in Germany (DAAD)

Lehrstuhl für Netzwerktheorie und Signalverarbeitung (NWS)
Institute for Circuit Theory and Signal Processing

- Head: Professor Josef A. Nossek
- Webpage: www.nws.ei.tum.de
- Research areas: Consistent modeling of physical layer
  - Processing of quantized signals
  - Cellular systems
  - Filter-bank based multicarrier systems
  - Cross-layer optimization
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  - Array processing for multipath and interference mitigation for Global Navigation Satellite Systems
  - Random matrix theory for wireless communications

Research collaborators-Some wonderful people!!

- Prof. Akbar M. Sayeed (University of Winsonsin-Madison)
- Prof. Peter J. Smith (University of Canterbury, New Zealand)
- Dr. Matthew R. McKay (Hong Kong University of Science and Technology, Hong Kong)
- Prof. George K. Karagiannidis (Aristotle University of Thessaloniki, Greece)
- Dr. David I. Laurenson (University of Edinburgh, UK)
- Dr. Cheng-Xiang Wang (University of Heriot-Watt, UK)

Presentation outline

- Overview of MIMO technology
- General applications of random matrix theory (RMT) to MIMO communications
- Generic framework for the standard condition number (SCN) distribution of Wishart matrices
- Conclusions
- Future research - Open problems

Overview of MIMO technology

- Developed by Telatar and Foschini in mid-90s
- Multiple antenna elements at both the transmitter/receiver
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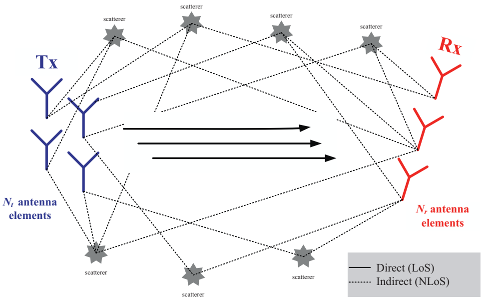

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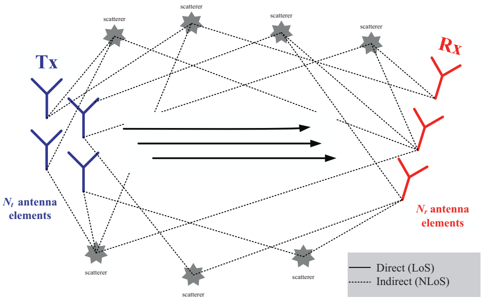

- High degree of multipath activity (i.i.d. Rayleigh fading) → linear capacity increase with $s = \min(N_t, N_r)$

Mathematical background

- MIMO system with N_t transmit and N_r receive antennas
- $s = \min(N_t, N_r)$ and $t = \max(N_t, N_r)$
- Complex input-output relationship

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{1}$$

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$$\mathbf{W} = \begin{cases} \mathbf{H}\mathbf{H}^{\dagger}, & \text{if } N_r \leq N_t \\ \mathbf{H}^{\dagger}\mathbf{H}, & \text{if } N_r > N_t \end{cases} \tag{2}$$

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is Hermitian, positive semi-definite and random.

Applications of RMT to MIMO communications - Example 1

- MIMO ergodic capacity assuming uniform power allocation

$$C = \mathcal{E}\left[\log_2\left(\det\left(\mathbf{I}_{N_r} + \frac{\text{SNR}}{N_t}\mathbf{H}\mathbf{H}^\dagger\right)\right)\right] \quad (3)$$

$$= \mathcal{E}\left[\sum_{k=1}^{s}\log_2\left(1 + \frac{\text{SNR}}{N_t}\lambda_k\right)\right] \quad (4)$$

$$= \int_0^\infty \log_2\left(1 + \frac{\text{SNR}}{N_t}\lambda\right)p(\lambda)d\lambda \quad (5)$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_s \geq 0$ are the real, non-negative, ordered eigenvalues of $\mathbf{W}$.

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$$

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- High-SNR ergodic capacity (SNR$\rightarrow \infty$)

$$
C = s\log_2(\mathrm{SNR}/N_t) + \frac{1}{\ln 2}\mathcal{E}\left[\ln(\det(\mathbf{H}\mathbf{H}^\dagger))\right] \qquad (6)
$$

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- Beamforming vector **w** with $E[|\mathbf{w}||^2] = 1$
- Transmitted symbol x, $E[|x||^2] = 1$
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- SNR at the output of combiner

$$\gamma = \bar{\gamma}\mathbf{w}^\dagger\mathbf{H}^\dagger\mathbf{H}\mathbf{w} \quad (9)$$

$$= \bar{\gamma}\mathbf{w}_{opt}^\dagger\mathbf{H}^\dagger\mathbf{H}\mathbf{w}_{opt} = \bar{\gamma}\lambda_1 \quad (10)$$

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Derive analytical closed-form expressions for the most important MIMO feautures

- Ergodic/outage capacity
- Higher-order capacity moments (variance, skewness,...)
- Symbol error rate (SER)-Outage probability of SM-MIMO
- Singular value decomposition (SVD) MIMO
- Performance of beamforming/MRC schemes
- Asymptotic characterization of MIMO systems (i.e. number of antennas grows infinitely large)

MIMO fading models and Wishart matrices

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- We need something simple and realistic at the same time!
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In any case, the correlation matrix **W** follows a complex Wishart distribution. This is some very good news for MIMO people!

MIMO fading models and Wishart matrices

Uncorrelated Rayleigh fading MIMO model

$$\mathbf{H} = \mathbf{H}_w \tag{11}$$

where $\mathbf{H}_w$ is an $(N_r \times N_t)$ matrix whose entries are complex zero-mean unity variance RVs, i.e. $\sim \mathcal{CN}(0, 1)$.

- The matrix $\mathbf{W}$ is uncorrelated central Wishart with t degrees of freedom (DoF), $\mathbf{W} \sim \mathcal{CW}_s(t, \mathbf{I}_s)$.

MIMO fading models and Wishart matrices

- Semi-correlated Rayleigh fading MIMO model

$$\mathbf{H} = \begin{cases} \boldsymbol{\Sigma}_s^{1/2}\mathbf{H}_w, & \text{if } N_r \leq N_t \\ \mathbf{H}_w \boldsymbol{\Sigma}_s^{1/2}, & \text{if } N_r > N_t. \end{cases} \quad (12)$$

with $\boldsymbol{\Sigma}_s \in \mathbb{C}^{s \times s}$ being a positive definite matrix containing the variances of the entries of **H** on its main diagonal.

- The matrix **W** is semi-correlated central Wishart with t DoF, $\mathbf{W} \sim \mathcal{CW}_s(t, \boldsymbol{\Sigma}_s)$.

MIMO fading models and Wishart matrices

- Uncorrelated Ricean fading MIMO model

$$\mathbf{H} = \underbrace{\sqrt{\frac{K_r}{K_r+1}}\mathbf{H}_{\mathrm{L}}}_{\text{LoS component}} + \underbrace{\sqrt{\frac{1}{K_r+1}}\mathbf{H}_w}_{\text{Scattered waves}} \tag{13}$$

where K_r is the Ricean K-factor.

- The matrix $\mathbf{W}$ is uncorrelated non-central Wishart with t DoF, $\mathbf{W} \sim \mathcal{CW}_s\left(t, 1/(K_r+1)\mathbf{I}_s, K_r/(K_r+1)\mathbf{H}_{\mathrm{L}}\mathbf{H}_{\mathrm{L}}^{\dagger}\right)$.

Investigation of the standard condition number (SCN)

- Ratio of the largest to the smallest eigenvalue

$$z = \frac{\lambda_1}{\lambda_s} \geq 1. \tag{14}$$

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- Applications in the area of wireless communications
  - Quantifies the performance of linear detectors (ZF, MMSE)
  - Adaptive MIMO transmission/Adaptive decoding
  - Level of multipath activity

Generic framework for the CDF of the SCN-part A

A1 Definition of the joint eigenvalue PDF

$$f(\boldsymbol{\lambda}) = K\,|\boldsymbol{\Phi}(\boldsymbol{\lambda})| \times |\boldsymbol{\Psi}(\boldsymbol{\lambda})| \prod_{\ell=1}^{s} \xi(\lambda_\ell). \tag{15}$$

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A2 Liebniz determinant formula (use all possible permutations of the matrix elements)

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A3 Joint eigenvalue PDF becomes

$$f(\boldsymbol{\lambda}) = K \sum_{\alpha} (-1)^{\alpha} |\psi_j(\lambda_i)\phi_{\alpha_i}(\lambda_i)\xi(\lambda_i)|, \; 1 \leq i,j \leq s. \qquad (17)$$

Generic framework for the CDF of the SCN-part B

B1 Integral definition of the SCN CDF, $F_z(x)$

$$F_z(x) = \Pr(z < x) = \int_0^{\infty} \left[\int_{\lambda_2}^{x\lambda_s} \int_{\lambda_3}^{x\lambda_s} \cdots \int_{\lambda_s}^{x\lambda_s} f(\lambda_1, \lambda_2, \ldots, \lambda_s) \right.$$

$$\left. d\lambda_{s-1} \ldots d\lambda_2 d\lambda_1 \right] d\lambda_s. \qquad (18)$$

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B2 Exploit the inherent symmetry of the joint eigenvalue PDF to permute the multiple integral $(s-1)$ times

$$F_z(x) = \frac{1}{(s-1)!} \int_0^\infty \int_{\lambda_s}^{x\lambda_s} \int_{\lambda_s}^{x\lambda_s} \cdots \int_{\lambda_s}^{x\lambda_s} f(\boldsymbol{\lambda}) d\lambda_1 \ldots d\lambda_{s-1} d\lambda_s \qquad (19)$$

Generic framework for the CDF of the SCN-part C

C1 After some crazy algebra, we end up with

$$F_z(x) = K \sum_{\ell=1}^{s} \int_0^{\infty} \left| \begin{bmatrix} \int_{\lambda_s}^{x\lambda_s} \phi_i(u)\psi_j(u)\xi(u)du, & i \neq \ell \\ \phi_i(\lambda_s)\psi_j(\lambda_s)\xi(\lambda_s), & i = \ell \end{bmatrix} \right| d\lambda_s$$

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C2 Closed-form expressions for $\int_{\lambda_s}^{x\lambda_s} \phi_i(u)\psi_j(u)\xi(u)du$

Eg. Uncorrelated central case:

$$\gamma(t - s + i + j - 1, x\lambda_s) - \gamma(t - s + i + j - 1, \lambda_s)$$

Analytical evaluation of the generic framework

- SCN CDF of an (8×3) MIMO system under uncorrelated Rayleigh, semi-correlated Rayleigh ($\rho = 0.6$) and uncorrelated Ricean ($K_r = 3$ dB) fading.

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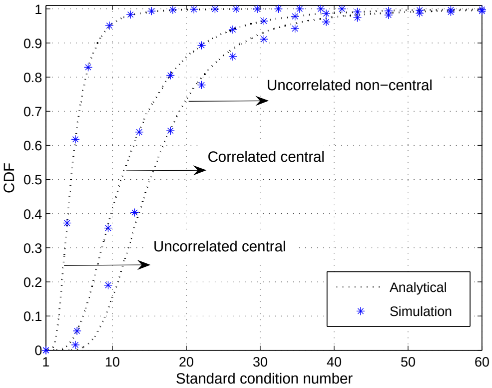
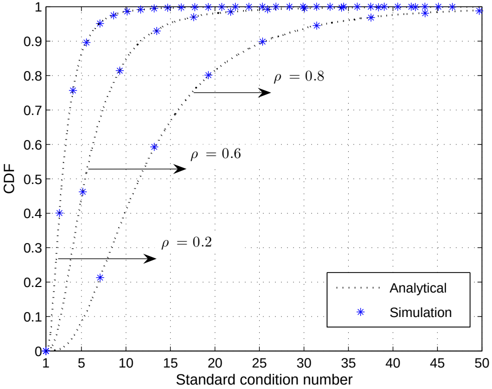


Simulation results

SCN CDF of a (6 $\times$ 2) semi-correlated Rayleigh system against the spatial correlation coefficient, ρ.



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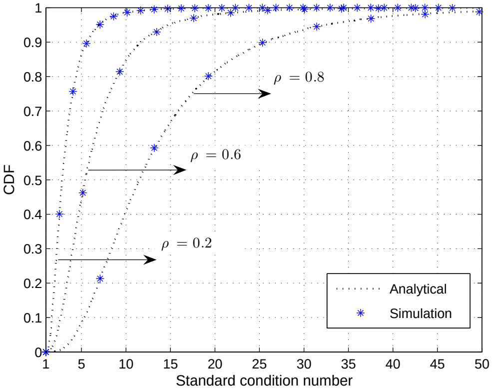
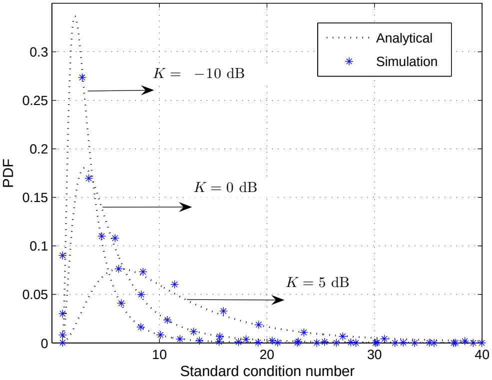


Spatial correlation $\rightarrow$ reduces the effective channel rank (identical spatial characteristics of the impigning multipaths).

Simulation results

SCN PDF of a (2×5) uncorrelated Ricean system against the Ricean K-factor



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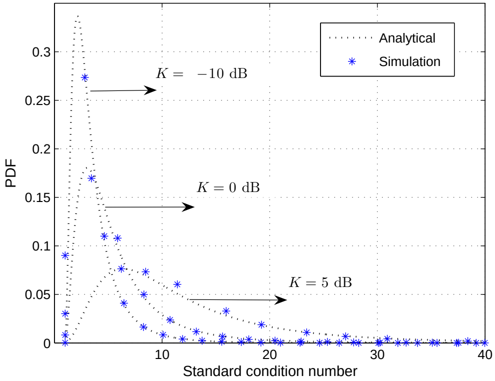

As K gets larger the dynamic range of the SCN increases too$\rightarrow$ excessive spatial correlation between the LoS rays' phases.

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- Validated via numerical integration for an arbitrary $(N_r \times N_t)$ MIMO system.
- Implications of the model parameters (spatial correlation, Ricean K-factor) on the condition number performance.

Future work - Open problems

- Application of RMT to cognitive radio characterization (e.g. spectrum sensing schemes)
- Extension of known RMT results from point-to-point MIMO to multiuser scenarios
- Design of practical adaptive subchannel selection algorithms for performance enhancement of MIMO beamforming schemes
- Performance analysis of MIMO systems assuming different fading models (e.g. Nakagami-*m*, Weibull...)
- Consider the impact of practical impairments (e.g. channel estimation error and feedback delay)

Related publications

- **M. Matthaiou**, M. R. McKay, P. J. Smith, and J. A. Nossek, “On the condition number distribution of complex Wishart matrices,” *in press IEEE Transactions on Communications*.
- **M. Matthaiou**, P. de Kerret, G. K. Karagiannidis, and J. A. Nossek, “Mutual information statistics and beamforming performance analysis of optimized LoS MIMO systems,” submitted to *IEEE Transactions on Communications*.
- **M. Matthaiou**, Y. Kopsinis, D. I. Laurenson, and A. M. Sayeed, “Upper bound for the ergodic capacity of dual MIMO Ricean systems: Simplified derivation and asymptotic tightness,” *IEEE Transactions on Communications*, vol. 57, no. 12, pp. pp. 3589–3596, December 2009.
- **M. Matthaiou**, D. I. Laurenson, and C. -X. Wang, “On analytical derivations of the condition number distributions of dual non-central Wishart matrices,” *IEEE Transactions on Wireless Communications*, vol. 8, no. 3, pp. 1212-1217, March 2009.
- **M. Matthaiou**, A. Pitarokoilis, and J. A. Nossek, “Mutual information statistics of optimized LoS MIMO systems,” to appear *IEEE International Conference on Communications (ICC)*, May 2010.

Thank you for your attention!